Regular subalgebras of affine Kac–Moody algebras

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Abstract

We classify regular subalgebras of Kac–Moody algebras in terms of their root systems. In the process, we establish that a root system of a subalgebra is always an intersection of the root system of the algebra with a sublattice of its root lattice. We also discuss applications to investigations of regular subalgebras of hyperbolic Kac–Moody algebras and conformally invariant subalgebras of affine Kac–Moody algebras. In particular, we provide explicit formulae for determining all Virasoro charges in coset constructions that involve regular subalgebras.

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1. Introduction

The main goal of this paper is to give a complete description of a large class of subalgebras of affine Kac–Moody algebras.

Recall that both untwisted and twisted Kac–Moody algebras afford a uniform description in terms of a combinatorial datum: roughly speaking, an algebra $\mathfrak{g}$ is formed by a Cartan subalgebra $\mathfrak{h}$ and root subspaces indexed by a discrete set Δ, called a root system. Our focus is on subalgebras whose root systems agree with that of $\mathfrak{g}$.

More precisely, consider a Kac–Moody algebra $\mathfrak{g}$. A subalgebra $\mathfrak{g}_1 \subset \mathfrak{g}$ is called *regular* if $\mathfrak{g}_1$ is invariant with respect to some Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. In other words, $\mathfrak{g}_1$ is a direct sum of a subspace of $\mathfrak{h}$ and subspaces of root spaces of $\mathfrak{g}$ (with respect to $\mathfrak{h}$). In terms of root systems, $\mathfrak{g}_1$ is regular if its root system Δ_1 is a specific subset of a root system Δ of $\mathfrak{g}$.

Our main result is the complete classification of regular affine subalgebras in both the untwisted and twisted case, see theorems 5.3, 5.6 and 5.8. We also describe a procedure

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for listing all regular subalgebras of affine Kac–Moody algebras. As a by-product, we give a classification of those regular subalgebras of hyperbolic Kac–Moody algebras whose generalized Cartan matrices are positive definite or semidefinite.

Our starting point is the classification of affine Kac–Moody algebras presented in [9, tables Aff 1–Aff 3] in terms of sets of simple roots of affine root systems. For basic facts concerning Kac–Moody algebras, we also refer the reader to [9]. In particular, we make use of the detailed description of affine root systems in [9, proposition 6.3].

Another important tool is the consideration of regular subalgebras from a geometric point of view. If $\mathfrak{g}_1 \subset \mathfrak{g}$ is a regular subalgebra, then the natural embedding of respective root systems $\Delta_1 \subset \Delta$ induces a natural embedding of the corresponding Weyl groups $W_1 \subset W$. Weyl groups of affine Kac–Moody algebras are exactly affine reflection groups or, in other words, discrete groups of isometries of a Euclidean space generated by reflections [9, proposition 3.13]. The relation with Weyl groups allows us, in turn, to use the results of [4], where all reflection subgroups of affine reflection groups were classified.

Subalgebras of Kac–Moody algebras appear in various contexts. In particular, just as an untwisted affine Kac–Moody algebra gives rise to a family of representations of the Virasoro algebra Vir (via the Sugawara construction), every pair subalgebra/algebra also gives rise to a family of representations of Vir. This is known as the coset construction. Its applications include, for example, the explicit construction of all irreducible unitary highest weight representations of Vir with charges between 0 and 1 [7]. Another application is the construction of consistent conformally invariant theories: such subalgebras allow string compactification while preserving conformal covariance. In this paper we restrict our attention to regular subalgebras only; however, in this case we are able to provide a recursive algorithm for determining all charges of Vir obtained via the coset construction. Since there is a formal analog of the coset construction for twisted affine algebras (see, e.g., [11]), we work with both untwisted and twisted algebras.

We note that our approach is not universal. Not every subalgebra $\mathfrak{g}_1$ of an affine Kac–Moody algebra $\mathfrak{g}$ is amenable to our (primarily combinatorial) methods: to be regular $\mathfrak{g}_1$ must be semisimple (e.g. this rules out the Heisenberg subalgebra) and its root system must agree with that of $\mathfrak{g}$ (this rules out certain embeddings obtained via representations of $\mathfrak{g}_1$). Nonetheless, we are able to deal effectively with a large and important class of subalgebras.

This paper is organized as follows. In section 2, we list essential facts concerning root systems and introduce notations used throughout this paper. In sections 3 and 4, we prove a necessary and sufficient condition for a pair of root systems $\Delta_1 \subset \Delta$ to define a pair algebra/subalgebra in terms of root lattices. Section 5 contains the classification of affine regular subalgebras of affine Kac–Moody algebras in terms of maximal subalgebras. This includes both affine and non-affine subalgebras. We simultaneously deal with both untwisted and twisted algebras. In section 6, we apply our results to the investigation of Virasoro charges obtained via the coset construction. We also describe all maximal conformally invariant regular subalgebras. Section 7 is devoted to subalgebras of hyperbolic Kac–Moody algebras.

2. Regular subalgebras and root subsystems

Denote by Δ a root system of an affine Kac–Moody algebra $\mathfrak{g}$. Our notations for the types of affine algebras follow the convention of [9, tables Aff 1–Aff 3]. We use the same notation for the types of corresponding root systems. Weyl groups are denoted as $\widetilde{X}_n$; we follow the convention of [16].

For groups and algebras of small ranks we will also use the following notation:

$$\widetilde{D}_2 = 2\widetilde{A}_1 \quad \widetilde{B}_1 = \widetilde{A}_1 \quad \widetilde{C}_1 = \widetilde{A}_1 \quad \widetilde{D}_3 = \widetilde{A}_3 \quad \widetilde{B}_2 = \widetilde{C}_2$$
$$D_2^{(1)} = 2A_1^{(1)} \quad B_1^{(1)} = \dot{A}_1^{(1)} \quad C_1^{(1)} = \ddot{A}_1^{(1)} \quad D_3^{(1)} = A_3^{(1)} \quad B_2^{(1)} = \dot{C}_2^{(1)} \quad A_1^{(2)} = \ddot{A}_1^{(1)}.$$

Here one dot indicates that the shortest root of the corresponding root system has length 1, while two dots indicate that the shortest root of the corresponding root system has length $\sqrt{2}$.

Recall that root systems of *untwisted* algebras are listed in table Aff 1; algebras of other types are called *twisted*.

By abuse of terminology, we will call a root system of an affine (respectively, hyperbolic) Kac–Moody algebra an *affine (hyperbolic) root system*. A root system is called *decomposable* if it is a union of two mutually orthogonal root systems. Otherwise it is called *indecomposable*. Indecomposable root systems correspond to simple algebras.

For a root system Δ we denote by Π a set of simple roots of Δ. We keep standard notations Δ^{re} and Δ^{im} for real and imaginary roots of Δ. By Δ_+ we mean the set of positive roots of Δ with respect to some fixed set of simple roots Π.

Let Δ be a root system of a Kac–Moody algebra $\mathfrak{g}$. Every regular subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$ has a root system $\Delta_1 \subset \Delta$ that is closed with respect to addition, i.e. satisfies the following condition:

$$\text{if } \alpha, \beta \in \Delta_1 \text{ and } \alpha + \beta \in \Delta, \text{ then } \alpha + \beta \in \Delta_1.$$

A root system $\Delta_1 \subset \Delta$ satisfying the condition above is called a *root subsystem* of Δ. (Another term used in the literature is subroot system.) It is easy to see that any root subsystem $\Delta_1 \subset \Delta$ is a root system of a regular subalgebra of $\mathfrak{g}$. Therefore, classifying regular subalgebras of $\mathfrak{g}$ is equivalent to classifying root subsystems of Δ.

Remark 2.1. We wish to emphasize the difference between Δ_1 and subsets of Δ used to construct representations of $\mathfrak{g}$. The latter are usually taken to contain a half of Δ. More precisely, the first step in constructing representations of $\mathfrak{g}$ is to take a set $P \subset \Delta$ closed with respect to addition such that $P \cup -P = \Delta$ (see, e.g., [5, 8]). Such a set P corresponds to a Borel or, more generally, parabolic subalgebra of $\mathfrak{g}$ whose representations yield those of $\mathfrak{g}$. On the other hand, we are looking for semisimple subalgebras of $\mathfrak{g}$. Thus, Δ_1 is a root system itself. For instance, unlike P, $\Delta_1 = -\Delta_1$.

To simplify checking if Δ_1 is a root subsystem of a root system Δ, we will restrict our attention to simple roots only. Specifically, we have the following explicit criterion for Δ_1 to be a root subsystem. Let Π_1 be a set of simple roots of Δ_1. Then Δ_1 is a root subsystem of Δ if and only if

$$\alpha - \beta \notin \Delta \text{ for all } \alpha, \beta \in \Pi_1. \tag{1}$$

This criterion is an immediate corollary of the following proposition.

Proposition 2.2 ([3, theorem 3.1])**.** *Let Δ be a root system of a Kac–Moody algebra $\mathfrak{g}$. Let $\beta_1, \ldots, \beta_k \in \Delta^{\text{re}}$ be positive real roots such that $\beta_i - \beta_j \notin \Delta$ for all $1 \leqslant i < j \leqslant k$. Let $\Delta' \subset \Delta$ be a minimal root system containing $\beta_1, \ldots, \beta_k$. Then Δ' is a root subsystem of Δ (and Δ' is a root system of a regular subalgebra of $\mathfrak{g}$).*

We say that a root subsystem $\Delta_1 \subset \Delta$ is *maximal* if for any root system $\Delta_2 \subset \Delta$ containing Δ_1 the system Δ_2 is not a subsystem of Δ. Clearly, maximal root subsystems correspond to maximal regular subalgebras.

3. Root subsystems in root systems

In this section, we provide several technical statements that will drastically reduce the computations necessary for classification.

We start with two elementary lemmas.

Lemma 3.1. *Let* $\Delta_2 \subset \Delta_1 \subset \Delta$ *be root systems.*

(1) If Δ_1 is a root subsystem of Δ and Δ_2 is a root subsystem of Δ_1, then Δ_2 is a root subsystem of Δ.
(2) If $\Delta_2 \subset \Delta_1$ is not a root subsystem, then $\Delta_2 \subset \Delta$ is also not a root subsystem.

Proof

(1) Let $\alpha, \beta \in \Delta_2$ and $\alpha + \beta \in \Delta$. To show that Δ_2 is a root subsystem of Δ, we check the definition by verifying that $\alpha + \beta \in \Delta_2$. Since $\Delta_2 \subset \Delta_1$, it follows that $\alpha, \beta \in \Delta_1$. By assumption, $\Delta_1 \subset \Delta$ is a root subsystem, which implies that $\alpha + \beta \in \Delta_1$. Since $\Delta_2 \subset \Delta_1$, we also have $\alpha + \beta \in \Delta_2$, so $\Delta_2 \subset \Delta$ is a root subsystem.
(2) By assumption there exist $\alpha, \beta \in \Delta_2$ such that $\alpha + \beta \in \Delta_1$ but $\alpha + \beta \notin \Delta_2$. Since $\Delta_1 \subset \Delta$, it follows that $\alpha + \beta \in \Delta$. Hence Δ_2 is not a root subsystem of Δ. □

Lemma 3.2. *Let Δ and $\Delta_1 \subset \Delta$ be root systems, and let Δ be simply-laced (i.e. all roots in Δ are of the same length). Then Δ_1 is a root subsystem of Δ.*

Proof. Let $\alpha, \beta \in \Pi_1$ be such that $\alpha - \beta \in \Delta$. Since α and β are simple roots, the angle formed by α and β is not acute, so $\alpha - \beta$ is longer than α or β, which is impossible.

□

Remark 3.3. The same argument as in the proof of lemma 3.2 also works in a non-simply-laced case. To check condition (1) in a non-simply-laced case, it is enough to check it for short roots α and β (and the roots of middle length in the case of $A_{2n}^{(2)}$).

Lemma 3.4 describes a procedure for classifying root subsystems $\Delta_1 \subset \Delta$ whose type coincides with the type of Δ.

Lemma 3.4. *Let Δ and $\Delta_1 \subset \Delta$ be root systems of affine algebras $\mathfrak{g}$ and $\mathfrak{g}_1$ of the same type and the same rank n. Let W and $W_1 \subset W$ be the corresponding Weyl groups. Then $[W : W_1] = k^n$ for some $k \in \mathbb{Z}_+$. Moreover,*

(1) If $\mathfrak{g}$ is untwisted, then $\Delta_1 \subset \Delta$ is a root subsystem for any k.
(2) If $\mathfrak{g}$ is a twisted algebra of a type different from $D_4^{(3)}$, then $\Delta_1 \subset \Delta$ is a root subsystem if and only if k is odd.
(3) If Δ is of type $D_4^{(3)}$, then $\Delta_1 \subset \Delta$ is a root subsystem if and only if k is not a multiple of 3.

Proof. It follows from [4, lemmas 9–11] that if W and $W_1 \subset W$ are Euclidean reflection groups of the same type, then either $[W : W_1] = k^n$ for some $k \in \mathbb{Z}_+$ or W is of one of the types $\widetilde{C}_2$, $\widetilde{G}_2$ and $\widetilde{F}_4$. In the latter case there exists a group W_2 of the same type as W such that $W_1 \subset W_2 \subset W$, $[W : W_2] = k^n$ for some k, and $[W_2 : W_1] = 2, 3$ or 4, respectively (depending on the type of W). Checking the groups $\widetilde{C}_2$, $\widetilde{G}_2$ and $\widetilde{F}_4$ directly, we see that the root systems corresponding to the groups W_2 and W_1 are of different types (their Dynkin diagrams differ by directions of arrows on the edges); however, the root systems corresponding to the groups W_2 and W have the same type. Therefore, the same type of Δ and Δ_1 implies $[W : W_1] = k^n$ for some $k \in \mathbb{Z}_+$.

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To determine which embeddings of root systems are root subsystems, we check condition (1). A subgroup W_1 is determined (up to an automorphism of W) by the type of W and the index $[W : W_1] = k^n$, so for each possible index we need to check one embedding only.

We will use an explicit presentation of affine root systems given in [9, proposition 6.3]. Denote by $\overset{\circ}{\Delta}$ the underlying finite root system of Δ (see [9, chapter 6]), and let $\overset{\circ}{\Delta}_s$ and $\overset{\circ}{\Delta}_l$ be the sets of short and long roots of $\overset{\circ}{\Delta}$, respectively.

(1) By [9, proposition 6.3], $\Delta = \{m\delta, \alpha + m\delta | \alpha \in \overset{\circ}{\Delta}, m \in \mathbb{Z}\}$. Then $\Delta_1 = \{mk\delta, \alpha + mk\delta | \alpha \in \overset{\circ}{\Delta}, m \in \mathbb{Z}\}$ where $k \in \mathbb{N}$. This implies that if $\alpha, \beta \in \Pi_1, \alpha - \beta \in \Delta$, then $\alpha - \beta \in \Delta_1$, which is impossible for simple roots α and β.

(2) If $\Delta \neq A_{2n}^{(2)}$, then [9, proposition 6.3] implies $\Delta = \{m\delta, \alpha + m\delta, \beta + 2m\delta | \alpha \in \overset{\circ}{\Delta}_s, \beta \in \overset{\circ}{\Delta}_l, m \in \mathbb{Z}\}$. Then $\Delta_1 = \{mk\delta, \alpha + mk\delta, \beta + 2mk\delta | \alpha \in \overset{\circ}{\Delta}_s, \beta \in \overset{\circ}{\Delta}_l, m \in \mathbb{Z}\}$ for some $k \in \mathbb{N}$. Note that Δ is a root system of a twisted algebra different from $A_{2n}^{(2)}$, thus Δ_1 contains two mutually orthogonal short simple roots α_1 and $-\beta_1 + k\delta$ (where $\alpha_1, \beta_1 \in \overset{\circ}{\Delta}_1$ and $\alpha_1 + \beta_1 \in \overset{\circ}{\Delta}_1$). So, if k is even, we have $\alpha_1 - (-\beta_1 + k\delta) \in \Delta$, and Δ_1 is not a root subsystem of Δ. If k is odd, then we check all pairs of short simple roots $\alpha, \beta \in \Pi_1$ and never obtain $\alpha - \beta \in \Delta$.

If $\Delta = A_{2n}^{(2)}$, then Π contains only one short simple root. So, if $\alpha, \beta \in \Pi_1$ and $\alpha - \beta \in \Delta_1$, then α and β are two orthogonal roots of the middle length. In this case $\alpha - \beta$ never belongs to Δ. Therefore, each root system $\Delta_1 \subset \Delta$ of type $A_{2n}^{(2)}$ is a root subsystem of Δ. On the other hand, if $W_1 \subset W$ are Weyl groups of type $\tilde{C}_n$ such that $[W : W_1] = k^n$ for an even k, then the root system Δ_1 consists of roots of only two different lengths. In this case Δ_1 is not a root system of type $A_{2n}^{(2)}$. In the case of odd $k\Delta_1$ contains roots of all three lengths, so it is a root system of type $A_{2n}^{(2)}$.

(3) The proof is similar to the proof of (2) for $\Delta \neq A_{2n}^{(2)}$. $\qquad\qquad\square$

4. Root subsystems and root lattices

The goal of this section is to prove theorem 4.4.

The following lemma holds for finite root systems as well as for affine and hyperbolic ones. The finite case is well known and the hyperbolic case may be found in [14]. Here we prove the affine case. For completeness we also include a proof of the finite case; see the claim below.

Lemma 4.1. *Let Δ and $\Delta_1 \subset \Delta$ be affine root systems of the same rank. Let L and $L_1 \subset L$ be the corresponding root lattices. Then Δ_1 is a root subsystem of Δ if and only if $\Delta_1 = \Delta \cap L_1$.*

Proof. First, assume that $\Delta_1 = \Delta \cap L_1$ but Δ_1 is not a root subsystem of Δ. Then there exist $\alpha, \beta \in \Pi_1$ such that $\alpha - \beta \in \Delta$. The condition $\Delta_1 = \Delta \cap L_1$ implies that $\alpha - \beta \in \Delta_1$. This is impossible since α and β are simple roots of Δ_1.

Now suppose Δ_1 be a root subsystem of Δ. Also suppose that $\Delta_1 \neq \Delta \cap L_1$, so that there exists $\alpha \subset \Delta$ such that $\alpha \in L_1 \backslash \Delta_1$.

Denote by $\overset{\circ}{\Delta}$ and $\overset{\circ}{\Delta}_1$ the underlying finite root systems of Δ and Δ_1, respectively (see [9, proposition 6.3]), and suppose that $\Delta_1 \neq A_{2n}^{(2)}$. Then L is generated over $\mathbb{Z}$ by $\overset{\circ}{\Delta}$ and δ, and L_1 is generated by $\overset{\circ}{\Delta}_1$ and $k\delta$ for some integer $k \geqslant 1$. Let $\alpha = \alpha_0 + l\delta, \alpha_0 \in \overset{\circ}{\Delta}$. Clearly, $\overset{\circ}{\Delta}_1$ is a root subsystem of $\overset{\circ}{\Delta}$. Denote by $\overset{\circ}{L}$ and $\overset{\circ}{L}_1 \subset \overset{\circ}{L}$ the root lattices of $\overset{\circ}{\Delta}$ and $\overset{\circ}{\Delta}_1$, respectively.

Claim. $\overset{\circ}{\Delta}_1 = \overset{\circ}{\Delta} \cap \overset{\circ}{L}_1$.

We will prove this claim for the case when $\overset{\circ}{\Delta}_1$ is a maximal root subsystem of $\overset{\circ}{\Delta}$. This will immediately imply the claim in the general case.

By the description of maximal regular subalgebras of semisimple Lie algebras (see [2, theorem 5.5]), in this case $\overset{\circ}{L}_1$ is a proper sublattice of $\overset{\circ}{L}$. The intersection $\widetilde{\Delta}_1 = \overset{\circ}{\Delta} \cap \overset{\circ}{L}_1$ is a root subsystem of $\overset{\circ}{\Delta}$ containing $\overset{\circ}{\Delta}_1$. Since $\overset{\circ}{\Delta}_1 \subset \overset{\circ}{\Delta}$ is maximal, we see that $\widetilde{\Delta}_1 = \overset{\circ}{\Delta}_1$.

Applying the claim above we have $\alpha_0 \in \overset{\circ}{\Delta}_1$. Moreover, $\alpha \in L_1$ implies $k \mid l$. Now, if Δ_1 is a root system of an untwisted algebra, we see that $\alpha \in \Delta_1$, contradicting our assumption. If Δ_1 is a root system of a twisted algebra and $\alpha \notin \Delta_1$, we see that α_0 is a long root of $\overset{\circ}{\Delta}_1$, and either $3 \nmid (l/k)$ (in the case of $\Delta_1 = D_4^{(3)}$) or l/k is odd (otherwise). In both the cases, at least one of $\alpha + k\delta$ and $\alpha - k\delta$ is a root of Δ_1. Combined with the assumption that $\alpha \in \Delta_1$ and $\pm k\delta \in \Delta_1$, this implies that Δ_1 is not a root subsystem of Δ. The contradiction proves the lemma for all $\Delta_1 \neq A_{2n}^{(2)}$.

In the case $\Delta_1 = A_{2n}^{(2)}$, we have $\Delta = A_{2n}^{(2)}$. Then $\overset{\circ}{\Delta}_1 = \overset{\circ}{\Delta}$. Furthermore, L is generated over $\mathbb{Z}$ by $\overset{\circ}{\Delta}$ and $(\alpha_n + \delta)/2$ (where α_n is a long root of $\overset{\circ}{\Delta}$), and L_1 is generated by $\overset{\circ}{\Delta}$ and $(\alpha_n + (2k-1)\delta)/2$ for some integer $k \geqslant 1$. Two cases are possible: either $\alpha = (\alpha_n + (2l-1)\delta)/2$ for a long root $\alpha_n \in \overset{\circ}{\Delta}$ and an integer l or $\alpha = \alpha_0 + l\delta$ for $\alpha_0 \in \overset{\circ}{\Delta}$ and an integer l.

Let $\alpha = (\alpha_n + (2l-1)\delta)/2$. Since $\alpha \in L_1$, we have $(2k-1) \mid (2l-1)$. This implies that $\alpha \in \Delta_1$.

Now let $\alpha = \alpha_0 + l\delta$. Again, $\alpha \in L_1$ implies $(2k-1) \mid (2l-1)$. Since $\alpha \notin \Delta_1$, we see that α_0 is a long root of $\overset{\circ}{\Delta}_1$ and $(2l-1)/(2k-1)$ is odd. Hence, $\alpha + (2k-1)\delta \in \Delta_1$. Furthermore, $(2k-1)\delta$ is also a root of Δ_1, which contradicts the assumption that Δ_1 is a root subsystem of Δ. This completes the proof of the lemma.

Remark 4.2. Note that to prove sufficiency of the condition $\Delta_1 = \Delta \cap L_1$, we do not need to assume that Δ_1 and Δ have the same rank.

Corollary 4.3. *Let Δ and $\Delta_1 \subset \Delta$ be affine root systems of the same rank. Let L and $L_1 \subset L$ be the corresponding root lattices, and let W and W_1 be the Weyl groups of Δ and Δ_1. Assume that W_1 is a maximal reflection subgroup of W. Then Δ_1 is a root subsystem of Δ if and only if L_1 is a proper sublattice of L.*

Moreover, this sublattice has index 2 unless $(\Delta_1, \Delta) = \left(A_2^{(1)}, G_2^{(1)}\right)$ or $\left(G_2^{(1)}, D_4^{(3)}\right)$. In the latter cases the index equals 3.

Proof. First, suppose that $L_1 = L$. Then $\Delta \cap L_1 = \Delta \cap L = \Delta \neq \Delta_1$. Lemma 4.1 implies that Δ_1 is not a root subsystem of Δ.

Now, let L_1 be a proper sublattice of L. By lemma 4.1 it suffices to check that $\Delta_1 = \Delta \cap L_1$. For Δ_1^{im} the assertion is obvious. Assume that there exists $\alpha \in \Delta^{\text{re}}$ such that $\alpha \in L_1$ such that $\alpha \notin \Delta_1$.

Consider the subgroup G_0 of W generated by reflections with respect to all roots in L_1. Obviously $W_1 \subset G_0$. Since W_1 is maximal in W and $G_0 \neq W_1$, it follows that $G = W_1$. Hence each simple root α in Δ can be represented as a linear combination of roots in L_1 with integer coefficients. Therefore α belongs to L_1 itself. Thus, $L = L_1$ contradicting the assumption.

A direct calculation shows that for each but two cases, the roots of the subsystem generate an index-two sublattice of the root lattice, while in the remaining cases the index of the sublattice equals 3. $\square$

To prove the main result of this section, we drop the assumption that Δ and its subsystem Δ_1 have the same rank.

Theorem 4.4. *Let* Δ *and* $\Delta_1 \subset \Delta$ *be affine root systems. Let* L *and* $L_1 \subset L$ *be the corresponding root lattices. Then* Δ_1 *is a root subsystem of* Δ *if and only if* $\Delta_1 = \Delta \cap L_1$.

Proof. We need to prove one implication only (see remark 4.2). Namely, suppose that Δ_1 is a root subsystem of Δ. Then we need to show that $\Delta_1 = \Delta \cap L_1$. Moreover, we may assume that $\Delta_1 \subset \Delta$ is a maximal root subsystem.

If ranks of Δ and Δ_1 are equal, then the statement follows from lemma 4.1. Hence, we can assume that the rank of Δ_1 is less than the rank of Δ. In particular, L_1 is a proper sublattice of L.

Consider the set $\Delta' = \Delta \cap L_1$. Clearly Δ' is a root subsystem of Δ containing Δ_1. Since L_1 does not coincide with L, Δ' is a proper subsystem of Δ. Now maximality of $\Delta_1 \subset \Delta$ implies $\Delta' = \Delta_1$ and the theorem is proved.

$\square$

5. Classification of regular subalgebras

Given an affine root system Δ_1 contained in an affine root system Δ, we can use condition (1) to determine if Δ_1 is a root subsystem of Δ. Our goal is to list all affine root subsystems of root systems of every affine Kac–Moody algebra.

Checking condition (1) is straightforward, and lemmas 3.1, 3.2 and 3.4 help to reduce the computations. Thus, essentially we only need to list all possible pairs (Δ, Δ_1). This is done as follows.

For each of the finitely many types of affine algebras (listed in [9, tables Aff 1–Aff 3]) we consider its root system Δ and Weyl group W. We list all possibilities for its reflection subgroup $W_1 \subset W$ using the results of [4]. Then for each subgroup $W_1 \subset W$ we check all possible root systems $\Delta_1 \subset \Delta$ corresponding to such a subgroup. For example, a subgroup of type $\widetilde{B}_n$ may correspond to a root system of either type $B_n^{(1)}$ or type $A_{2n-1}^{(2)}$, while a subgroup of type $\widetilde{C}_n$ may correspond to one of the types $C_n^{(1)}$, $D_{n+1}^{(2)}$ and $A_{2n}^{(2)}$. Furthermore, we consider not only the types of subgroups (and the corresponding root systems) but also all different embeddings of subgroups of this type. Most embeddings of Euclidean reflection subgroups of finite index of type $\widetilde{X}$ in Euclidean reflection groups of type $\widetilde{Y}$ can be distinguished by the indices of subgroups. So, to indicate which embeddings correspond to subalgebras we specify the indices of respective subgroups (in the case of finite index). If a subgroup is of infinite index, we check only if a root system Δ_1 of a given type can be a root subsystem of Δ. (If the answer is affirmative, we place this subsystem in the resulting tables without further distinguishing it from other embeddings, which may not be root subsystems.) For some types of root systems Δ and $\Delta_1 \subset \Delta$, the system Δ_1 may be either short or long (for example, for $D_4^{(1)} \subset F_4^{(1)}$). In such cases we check all possibilities and state in the tables below whether the root subsystem is short or long.

Remark 5.1. It was stated in [4, theorem 1] that if W is an indecomposable group of a given type Y, then all reflection subgroups of a given type X can be distinguished by their indices (modulo the automorphism group of W). However, while preparing this paper, we discovered an exception to this rule.

Namely, if W and W_1 are of types $\widetilde{F}_4$ and $\widetilde{C}_4$, respectively, there are two different embeddings of a subgroup of type $\widetilde{C}_4$ into a group of type $\widetilde{F}_4$ with the same index $[W : W_1] = 24k^4$ for any $k \in \mathbb{N}$. Specifically, there is a chain of embeddings $\widetilde{C}_4 \subset \widetilde{B}_4 \subset \widetilde{F}_4$,

where an automorphism of $\widetilde{C}_4$ may be extended to an automorphism of $\widetilde{B}_4$, but not to an automorphism of $\widetilde{F}_4$.

The oversight resulted from an omission in a case-by-case check at the very end of the proof of theorem 1.

To make our tables more readable, we list maximal root subsystems only (others can easily be retrieved from the lists of maximal ones). Note that a decomposable maximal root subsystem does not necessarily correspond to a maximal subgroup, namely, for a chain of root systems $\Delta_1 \subset \Delta_2 \subset \Delta_3$ it is possible that $\Delta_2 \subset \Delta_3$ is not a root subsystem, while $\Delta_1 \subset \Delta_3$ is.

First, we consider indecomposable affine root subsystems of maximal rank, so that $W_1 \subset W$ is an indecomposable Euclidean reflection subgroup of finite index. Then we allow decomposable W_1. Further on, we move to affine root subsystems of lower rank (and then $[W : W_1]$ is infinite). Finally, we describe remaining root subsystems that correspond to direct products of a spherical and a Euclidean reflection group.

5.1. Indecomposable affine root subsystems of maximal rank

Below we provide a typical procedure of classifying maximal rank root subsystems of the root system of an untwisted algebra. Dealing with twisted algebras usually involves more computations but is not otherwise different.

Example 5.2 (indecomposable affine root subsystems of $B_n^{(1)}, n \geqslant 3$). The Weyl group W is of the type $\widetilde{B}_n$. The results in [4] imply that its indecomposable finite index subgroups have types $\widetilde{B}_n, \widetilde{C}_n$ and $\widetilde{D}_n$. For subgroups of these types we should consider root systems of types $B_n^{(1)}$ and $A_{2n-1}^{(2)}$ (for $\widetilde{B}_n$), $C_n^{(1)}, D_{n+1}^{(1)}$ and $A_{2n}^{(2)}$ (for $\widetilde{C}_n$) and $D_n^{(1)}$ (for $\widetilde{D}_n$). The root system of type $A_{2n}^{(2)}$ is irrelevant because it contains roots of three different lengths, while $B_n^{(1)}$ only has roots of two different lengths. Furthermore, short roots of $B_n^{(1)}$ are mutually orthogonal, which implies that neither $A_{2n-1}^{(2)}$ nor $C_n^{(1)}$ can be contained in $B_n^{(1)}$. So, we need to check only root systems of types $B_n^{(1)}, D_{n+1}^{(2)}$ and $D_n^{(1)}$. Note that embeddings of groups $\widetilde{B}_n, \widetilde{C}_n$ and $\widetilde{D}_n$ into $\widetilde{B}_n$ are determined by indices up to isomorphisms of $\widetilde{B}_n$, so it suffices to check one embedding for each possible index. Explicit check shows that embeddings of $B_n^{(1)}$ and $D_n^{(1)}$ satisfy condition (1), while that of $D_{n+1}^{(2)}$ does not.

Theorem 5.3. *Let Δ be an indecomposable affine root system and $\Delta_1 \subset \Delta$ a maximal root subsystem. If Δ_1 is an indecomposable affine root subsystem of maximal rank, then $\Delta_1 \subset \Delta$ is one of the root subsystems listed in table 1.*

Remark 5.4. Our notation in tables 1 and 2 is as follows. If a root system of some type $X_n^{(r)}$ may be embedded in Δ with different root lengths, we write $\dot{X}_n^{(r)}$ for a 'short' embedding and $\ddot{X}_n^{(r)}$ for the 'long' one.

5.2. Decomposable affine root subsystems of maximal rank

Example 5.5 (decomposable affine root subsystems of $B_n^{(1)}, n \geqslant 3$). As shown in example 5.2, $B_n^{(1)}$ may contain root systems of types $B_n^{(1)}, D_n^{(1)}$ and $D_{n+1}^{(2)}$ (not necessary as root subsystems). Furthermore, it follows from [4, table 5] that any maximal subgroup of $\widetilde{B}_n$ is $\widetilde{B}_m + \widetilde{B}_{n-m}$ ($m \geqslant 1$), and similarly maximal subgroups of $\widetilde{C}_n$ and $\widetilde{D}_n$ are $\widetilde{C}_m + \widetilde{C}_{n-m}$ and $\widetilde{D}_m + \widetilde{D}_{n-m}$, respectively. So, looking through the chains of maximal subgroups (and

Table 1. Indecomposable maximal affine root subsystem of maximal rank. The right column lists possible indices of Weyl subgroups $W_1 \subset W$ corresponding to maximal root subsystems $\Delta_1 \subset \Delta$ (p is any prime number).

Δ	Δ_1	$[W:W_1]$	Δ	Δ_1	$[W:W_1]$
$A_1^{(1)}$	$A_1^{(1)}$	p	$A_2^{(2)}$	$A_2^{(2)}$	$p,\ p \neq 2$
$A_n^{(1)}$ ($n \geq 2$)	$A_n^{(1)}$	p^n		$\dot{A}_1^{(1)}$	2
$B_n^{(1)}$ ($n \geq 3$)	$B_n^{(1)}$	p^n		$\ddot{A}_1^{(1)}$	2
	$D_n^{(1)}$	2	$A_{2n}^{(2)},\ n \geq 2$	$A_{2n}^{(2)}$	$p^n,\ p \neq 2$
$C_n^{(1)}$ ($n \geq 2$)	$C_n^{(1)}$	p^n		$A_{2n-1}^{(2)}$	2
$D_n^{(1)}$ ($n \geq 4$)	$D_n^{(1)}$	p^n		$B_n^{(1)}$	2^{n+1}
$G_2^{(1)}$	$\ddot{A}_2^{(1)}$	2	$A_4^{(2)}$	$D_3^{(2)}$	2
	$G_2^{(1)}$	p^2	$A_{2n-1}^{(2)}\ n \geq 3$	$A_{2n-1}^{(2)}$	$p^n,\ p \neq 2$
$F_4^{(1)}$	$B_4^{(1)}$	3		$C_n^{(1)}$	2^{n-1}
	$F_4^{(1)}$	p^4	$D_{n+1}^{(2)}\ n \geq 2$	$D_{n+1}^{(2)}$	$p^n,\ p \neq 2$
$E_6^{(1)}$	$E_6^{(1)}$	p^6		$B_n^{(1)}$	2
$E_7^{(1)}$	$A_7^{(1)}$	$2^4 \times 3^2$	$D_3^{(2)}$	$C_2^{(1)}$	2
	$E_7^{(1)}$	p^7	$E_6^{(2)}$	$E_6^{(2)}$	$p^4,\ p \neq 2$
$E_8^{(1)}$	$A_8^{(1)}$	$2^7 \times 3^2 \times 5$		$F_4^{(1)}$	4
	$D_8^{(1)}$	$2 \times 3^3 \times 5$		$C_4^{(1)}$	$2^3 \times 3$
	$E_8^{(1)}$	p^8	$D_4^{(3)}$	$D_4^{(3)}$	$p^2,\ p \neq 3$
				$G_2^{(1)}$	3
				$\dot{A}_2^{(1)}$	2

taking into account that short roots of $B_n^{(1)}$ are mutually orthogonal) we conclude that any root subsystem Δ_1 of $B_n^{(1)}$ splits into components of types $B_m^{(1)}$, $D_m^{(1)}$ and $D_{m+1}^{(2)}$ (with possibly different ranks of components). If Δ_1 has at least one component of type $D_{m+1}^{(2)}$, then Δ_1 is not a root subsystem. Indeed, if α and β are short simple roots of $D_{m+1}^{(2)}$, then $\alpha - \beta$ belongs to $B_n^{(1)}$. Similarly, if Δ_1 is a root subsystem, then it contains at most one component of type $B_m^{(1)}$. Therefore, any decomposable affine root subsystem (of maximal rank) in $B_n^{(1)}$ can be written as $\varepsilon B_{m_0}^{(1)} + D_{m_1}^{(1)} + \cdots + D_{m_s}^{(1)}$, where $\varepsilon = 0$ or 1 and $\varepsilon m_0 + m_1 + \cdots + m_s = n$ for $m_0 \geq 1$ and $m_i \geq 2, i = 1, \ldots, s$. Clearly, all these root systems are root subsystems of $\widetilde{B}_n$, and the only maximal subsystems are $B_m^{(1)} + D_{n-m}^{(1)}$ for $m = 1, \ldots, n-2$.

Similar straightforward calculations for every indecomposable affine root system imply the following theorem.

Theorem 5.6. *Let Δ be an indecomposable affine root system and $\Delta_1 \subset \Delta$ its maximal root subsystem. If Δ_1 is a decomposable root subsystem of maximal rank, then $\Delta_1 \subset \Delta$ is one of the subsystems listed in table 2.*

Table 2. Decomposable maximal affine root subsystems of maximal rank.

Δ	Δ_1	$[W:W_1]$	Δ	Δ_1	$[W:W_1]$
$B_n^{(1)}$	$B_m^{(1)} + D_{n-m}^{(1)}$	$4\binom{n}{m}$	$A_{2n-1}^{(2)}$	$A_{2m-1}^{(2)} + A_{2n-2m-1}^{(2)}$	$2\binom{n}{m}$
$C_n^{(1)}$	$C_m^{(1)} + C_{n-m}^{(1)}$	$\binom{n}{m}$	$A_{2n}^{(2)}$	$A_{2m}^{(2)} + A_{2n-2m-1}^{(2)}$	$2\binom{n}{m}$
$D_n^{(1)}$	$D_m^{(1)} + D_{n-m}^{(1)}$	$4\binom{n}{m}$		$D_m^{(1)} + A_{2n-2m}^{(2)}$	$2^{m+2}\binom{n}{m}$
$E_6^{(1)}$	$A_5^{(1)} + A_1^{(1)}$	$2^3 \times 3^2$		$\ddot{D}_3^{(2)} + A_{2n-4}^{(2)}$	$2\binom{n}{m}$
	$3A_2^{(1)}$	$2^4 \times 3^2 \times 5$	$D_{n+1}^{(2)}$	$D_{n-m}^{(1)} + D_{m+1}^{(2)}$	$4\binom{n}{m}$
$E_7^{(1)}$	$D_6^{(1)} + A_1^{(1)}$	$2 \times 3^2 \times 7$	$E_6^{(2)}$	$A_5^{(2)} + \ddot{A}_1^{(1)}$	$2^3 \times 3$
	$A_5^{(1)} + A_2^{(1)}$	$2^5 \times 3^2 \times 7$	$D_4^{(3)}$	$\dot{A}_1^{(1)} + \ddot{A}_1^{(1)}$	6
$E_8^{(1)}$	$E_7^{(1)} + A_1^{(1)}$	$2^4 \times 3 \times 5$			
	$E_6^{(1)} + A_2^{(1)}$	$2^6 \times 3 \times 5 \times 7$			
	$2A_4^{(1)}$	$2^8 \times 3^3 \times 5 \times 7$			
$F_4^{(1)}$	$\dot{A}_2^{(1)} + \ddot{A}_2^{(1)}$	$2^5 \times 3$			
	$C_3^{(1)} + \ddot{A}_1^{(1)}$	$2^7 \times 3$			
$G_2^{(1)}$	$\dot{A}_1^{(1)} + \ddot{A}_1^{(1)}$	6			

5.3. Affine root subsystems of lower rank

To classify root subsystems of lower rank we use the same procedure as in the case of maximal rank. More precisely, we consider chains of subgroups $G_1 \subset G_2 \subset \cdots \subset G_s = W$, where W is a Weyl group of $\mathfrak{g}$ and each G_i a maximal subgroup of G_{i+1}. Then we check which root systems correspond to G_i and which of those are root subsystems. Since we obtain root subsystems of ranks smaller than that of $\mathfrak{g}$, one of the subgroups $G_i \subset G_{i+1}$ must be a maximal subgroup of infinite index. Such subgroups are described in [4, theorem 3]. Namely, these maximal subgroups are of the following types: $\widetilde{A}_k + \widetilde{A}_{n-1-k} \subset \widetilde{A}_n$ $(k \leqslant n-1)$; $\widetilde{D}_{n-1} \subset \widetilde{D}_n$; $\widetilde{A}_{n-1} \subset \widetilde{D}_n$; $\widetilde{D}_5 \subset \widetilde{E}_6$ and $\widetilde{E}_6 \subset \widetilde{E}_7$.

Example 5.7 (lower rank affine root subsystems of $B_n^{(1)}$, $n \geqslant 3$)**.** As in example 5.5 we see that any affine root subsystem of $B_n^{(1)}$ can be written as $\varepsilon B_{m_0}^{(1)} + D_{m_1}^{(1)} + \cdots + D_{m_s}^{(1)}$, where $\varepsilon = 0$ or 1. If $\varepsilon m_0 + m_1 + \cdots + m_s < n$, then this is a root subsystem of $B_{n-1}^{(1)}$, which in its turn is a root subsystem of $B_n^{(1)}$. So, the only maximal root subsystem of lower rank is $B_{n-1}^{(1)}$.

Subgroups described in [4, theorem 3] do not affect the list of maximal root subsystems of $B_n^{(1)}$. Indeed, among the groups $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$ and $\widetilde{E}_7$, only $\widetilde{D}_n$ is a subgroup of $\widetilde{B}_n$. The corresponding root system $D_n^{(1)}$ is a root subsystem of $B_n^{(1)}$, thus none of the root subsystems of $D_n^{(1)}$ can be maximal in $B_n^{(1)}$.

Applying the above procedure to each indecomposable affine root systems, we obtain the following theorem.

Theorem 5.8. *Let Δ be an indecomposable affine root system and $\Delta_1 \subset \Delta$ a maximal affine root subsystem. If the rank of Δ_1 is less than the rank of Δ, then (Δ, Δ_1) is one of the following pairs:*

$\left(A_n^{(1)}, A_k^{(1)} + A_{n-1-k}^{(1)}\right)$, 　　　$\left(E_7^{(1)}, E_6^{(1)}\right)$, 　　　$\left(B_n^{(1)}, B_{n-1}^{(1)}\right)$, 　　　$\left(E_6^{(1)}, D_5^{(1)}\right)$,

$\left(C_n^{(1)}, A_{n-1}^{(1)}\right)$, 　　　$\left(A_{2n-1}^{(2)}, A_{n-1}^{(1)}\right)$, 　　　$\left(D_n^{(1)}, A_{n-1}^{(1)}\right)$, 　　　$\left(D_{n+1}^{(2)}, D_n^{(2)}\right)$.

5.4. Non-affine root subsystems

Let Δ be an affine root system. Consider a regular subsystem $\Delta_1 \subset \Delta$. Clearly Δ_1 is a direct sum of an affine and a finite root system Δ_a and Δ_s. Let $\Delta_s = \Delta_1 \cup \cdots \cup \Delta_k$, where Δ_i are indecomposable and mutually orthogonal. Let Π_i be a set of simple roots of Δ_i, $1 \leqslant i \leqslant k$. Denote by $\Pi_s = \Pi_1 \cup \cdots \cup \Pi_k$ a set of simple roots of Δ_s. Add to each Π_i a root $\beta_i = \theta_i + k_i \delta$, where θ_i is the lowest root of Δ_i and k_i is the least positive integer such that $\theta_i + k_i \delta \in \Delta$. Then $\Pi_a \stackrel{\text{def}}{=} \bigcup_{i=1}^{s} \{\Pi_i, \beta_i\}$ is a set of simple roots of an affine root subsystem of Δ that contains Δ_1.

Reversing the procedure above, we obtain the following theorem.

Theorem 5.9. *Let Δ be an affine root system and $\widetilde{\Delta}$ its affine root subsystem. Let $\Delta_1, \ldots, \Delta_k$ be indecomposable components of $\widetilde{\Delta}$. Let Π be a set of simple roots of $\widetilde{\Delta}$ and Π_i a set of simple roots of Δ_i, $1 \leqslant i \leqslant k$. Denote by $J \subset \{1, \ldots, k\}$ the set of indices such that $j \in J$ if Δ_j is a root system of an untwisted algebra. Let J' be a subset of J. For each $j \in J'$ discard the root $\alpha_0^j \in \Pi_j$ from Π (see [9, table Aff 1]) and denote by Π' the set of simple roots obtained. Let $\Delta' \subset \Delta$ be the root system whose set of simple roots is Π'.*

Then Δ' is a root subsystem of Δ.

Moreover, for every non-affine root subsystem Δ' of Δ, there exists an affine root subsystem $\widetilde{\Delta}$ of Δ such that Δ' can be obtained from $\widetilde{\Delta}$ in this way.

6. Conformally invariant regular subalgebras of affine Kac–Moody algebras

Recall that an untwisted affine Kac–Moody algebra is an extension of a loop algebra over a semisimple finite-dimensional Lie algebra $\mathfrak{g}$. For the duration of this section, we fix the notation $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$, where K is a central element. The bracket in $\hat{\mathfrak{g}}$ is defined as $[at^m, bt^n] = [a, b]t^{m+n} + \delta_{m,-n}(a|b)K$, where $(|)$ is an invariant form on $\mathfrak{g}$ such that the square of the highest root of $\mathfrak{g}$ is 2. Given an irreducible representation of $\hat{\mathfrak{g}}$, K necessarily acts as a scalar k called the level of this representation. Though k can take on any value, in physical literature it is assumed to be integral and positive. For the most part we will impose some of these restrictions as well. However, it should be pointed out that from a purely mathematical standpoint half-integer levels are also interesting (see, e.g., [13] for a vertex algebra construction related to a pair of conformally invariant algebras discussed below).

It is well known that the vacuum module $V_k(\mathfrak{g})$ of $\hat{\mathfrak{g}}$ of level k carries the structure of a vertex operator algebra with the canonical conformal vector $\omega_k(\mathfrak{g})$ given by the Sugawara construction. The central charge of $\omega_k(\mathfrak{g})$ is

$$D_k(\mathfrak{g}) = \sum \frac{k \dim \mathfrak{g}_i}{k + \check{h}_i},$$

where $\mathfrak{g}_i$ are irreducible components of $\mathfrak{g}$, $\mathfrak{g} = \oplus \mathfrak{g}_i$ and $\check{h}_i$ is the dual Coxeter number of $\mathfrak{g}_i$. The Fourier coefficients of $\omega_k(\mathfrak{g})$ form a Virasoro algebra that acts naturally on $\hat{\mathfrak{g}}$-modules of level k [10]. We can thus construct a large family of representations of the Virasoro algebra with central charge $D_k(\mathfrak{g})$ with desired properties (unitary, of highest weight, etc).

6.1. Coset construction

This construction can be generalized via the so-called coset construction as follows [6, 10]. Consider a graded subalgebra $\hat{\mathfrak{h}} \subset \hat{\mathfrak{g}}$ whose Cartan subalgebra is in the Cartan subalgebra of $\hat{\mathfrak{g}}$ and has the same imaginary root for some choice of root system. (Apart from subalgebras of the same type, all regular subalgebras constructed above satisfy this condition.) $V_{k'}(\mathfrak{h})$ naturally embeds into $V_k(\mathfrak{g})$ for a specific choice of k'. Namely, for $\hat{\mathfrak{h}}$ simple, let j be the square of the

highest root in $\hat{\mathfrak{h}}$ with respect to the form $(\,|\,)$ on $\hat{\mathfrak{g}}$. Then $k' = 2k/j$. (The quantity $2/j$ is known as the Dynkin index of $\mathfrak{h}$ in $\mathfrak{g}$.) In the semisimple case, $V_{k'}(\mathfrak{h})$ is a direct sum of affine vertex algebras corresponding to simple components of $\mathfrak{h}$. We can thus consider the difference $\omega_k(\mathfrak{h}, \mathfrak{g}) = \omega_k(\mathfrak{g}) - \omega_k(\mathfrak{h})$. A direct computation utilizing the Sugawara construction shows that $\omega_k(\mathfrak{h}, \mathfrak{g})$ is conformal with the central charge $D_k(\mathfrak{g}, \mathfrak{h}) = D_k(\mathfrak{g}) - D_k(\mathfrak{h})$.

If $D_k(\mathfrak{g}, \mathfrak{h}) \neq 0$, we obtain yet another construction of representations of the Virasoro algebra with a given charge. In particular, this strategy was used in the seminal work [7] to construct all irreducible unitary highest weight representations of the Virasoro algebra with charges between 0 and 1.

The case $D_k(\mathfrak{g}) = D_k(\mathfrak{h})$ is also important, since then the pair $\mathfrak{h} \subset \mathfrak{g}$ can be used to construct consistent conformally invariant theories. Such pairs were studied in, e.g., [1] or [15]. Both papers imposed certain restrictions such as $k = 1$ (or, more generally, positive integral) in [1] or algebras being simply-laced in [15].

Below, using our classification of regular subalgebras of simple non-twisted Kac–Moody algebras, we provide a unified strategy for computing all possible values of central charge for $\omega_k(\mathfrak{h}, \mathfrak{g})$, where $\mathfrak{h}$ is a regular subalgebra of $\mathfrak{g}$. We also indicate when $\mathfrak{h}$ is conformally invariant (i.e. when $D_k(\mathfrak{g}) = D_k(\mathfrak{h})$).

Note first that if $\hat{\mathfrak{h}}$ is not a maximal subalgebra of $\hat{\mathfrak{g}}$, i.e. if $\hat{\mathfrak{h}} \subset \hat{\mathfrak{h}}' \subset \hat{\mathfrak{g}}$, then it suffices to compute the required values for the pairs $\hat{\mathfrak{h}} \subset \hat{\mathfrak{h}}'$ and $\hat{\mathfrak{h}}' \subset \hat{\mathfrak{g}}$. Thus, we restrict our attention to maximal subalgebras only. As noted above, we will also consider only subalgebras of the type different from $\hat{\mathfrak{g}}$'s. Moreover, if $\mathfrak{g}$ is not simple, i.e. $\mathfrak{g} = \oplus \mathfrak{g}_i$, then it suffices to consider projections of $\mathfrak{h}$ onto $\mathfrak{g}_i$ as subalgebras of $\mathfrak{g}_i$. Thus, we can assume that $\mathfrak{g}$ is simple.

6.2. Conformally invariant subalgebras of non-twisted algebras

We consider each of the types of $\mathfrak{g}$ in turn. The values of dual Coxeter numbers for $\hat{\mathfrak{g}}$ can be found in, e.g., [9, section 6.1]. We assume that $k \neq \check{h}$ (i.e. that the level k is not critical) for all algebras and subalgebras considered.

A_n. A maximal regular subalgebra $\mathfrak{h}$ must be of the type $A_m + A_{n-m-1}, 0 < m < n$. (Here and below we freely use subalgebra lists in theorems 5.3, 5.6 and 5.8.) Then $D_k(\mathfrak{g}, \mathfrak{h}) = k\{(n + 2k + 1)(n + k(n - m)(m + 1)) - m(n - m - 1) + (k + 1)^2/(k + n + 1)(k + m + 1)(k + n - m)\}$. Clearly for $k, n, m > 0, n + 2k + 1 > m$ and $n + k(n - m)(m + 1) > n - m - 1$, hence the numerator in $D_k(\mathfrak{g}, \mathfrak{h})$ is positive. It follows that there are no conformally invariant maximal regular subalgebras for k positive. More generally, a regular subalgebra of $\mathfrak{g}$ of type A_n is a direct sum of subalgebras of the same type. Thus, it suffices to consider a subalgebra $\mathfrak{h}$ of type A_m. Here $D_k(\mathfrak{g}, \mathfrak{h}) = k\{(n - m)((n + m + 2)(k + 1) + nm)/(k + n + 1)(k + m + 1)\}$. Thus, there are no conformally invariant subalgebras in type A_m for k positive.

B_n. For $\mathfrak{h}$ of type $B_{n-1}, D_k(\mathfrak{g}, \mathfrak{h}) = k\{(4n^2 - 8n + 4kn - k + 1)/(k + 2n - 1)(k + 2n - 3)\}$. The numerator, if viewed as a function of n, has discriminant $k^2 - 3k + 3$. For an integer k, it is a full square only for $k = 2$. Thus, the numerator is never zero if k and n are integral and there are no conformally invariant pairs $(\mathfrak{g}, \mathfrak{h})$ of this type.

For $\mathfrak{h}$ of type $D_n, D_k(\mathfrak{g}, \mathfrak{h}) = kn\{(2k + 2n - 3)/(k + 2n - 1)(k + 2(n - 1))\}$. Clearly, there are no conformally invariant pairs for any n on every integer level.

For a decomposable subalgebra $\mathfrak{h}$ of type $B_m + D_{n-m}, 0 < m < n, D_k(\mathfrak{g}, \mathfrak{h}) = k(n - m)(2m + 1)\{(k - 1)(2n + 2k - 3)/(k + 2n - 1)(k + 2m - 1)(k + 2(n - m) - 2)\}$. The subalgebra is conformally invariant on level 1. Note that just as in the case of a subalgebra of type D_n, we also get a conformally invariant subalgebra on the same half-integer level $-n - 3/2$.

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C_n. For $\mathfrak{h}$ of type A_{n-1}, $D_k(\mathfrak{g}, \mathfrak{h}) = k\{(n^3 + n^2k + 2kn + k + n + 1)/(k + n + 1)(2k + n)\}$. Setting the numerator to zero and solving for integral k, we see that $n \equiv 1 \bmod 3$. Also, a direct computation shows that $-n/3 < k < 1 - n/3$. Both statements imply that $k = n/3 - 1$ but for such a value of k the numerator is zero only when $n = -2$. Hence there are no conformally invariant pairs of this type.

If $\mathfrak{h}$ is decomposable of type $C_m + C_{n-m}, 0 < m < n$, then $D_k(\mathfrak{g}, \mathfrak{h}) = m(n - m)k\{(2k + 1)(2k + n + 2)/(k + n + 1)(k + m + 1)(k + n - m + 1)\}$. Hence $\mathfrak{h}$ is conformally invariant only for even n on the level $k = -n/2 - 1$. Also, if $\mathfrak{h}$ is indecomposable of type C_m, $D_k(\mathfrak{g}, \mathfrak{h}) = k(n - m)\{(2(m + n) + 1)(k + 1) + 2mn/(k + n + 1)(k + m + 1)\}$. For k positive, there are no conformally invariant subalgebras of this type.

D_n. For $\mathfrak{h}$ of type A_{n-1}, $D_k(\mathfrak{g}, \mathfrak{h}) = k\{(k+1)n^2 + (2-k)n + k - 2/(k + 2(n - 1))(k + n)\}$. The numerator, if viewed as a function of n, has discriminant $12 - 3k^2$. For an integer k, it is a full square only for $k = 1, 2$, at which levels the numerator is not zero. Thus, $D_k(\mathfrak{g}, \mathfrak{h}) \neq 0$ for any integral k and $n > 1$ and there are no conformally invariant pairs of this type.

For a decomposable maximal subalgebra $\mathfrak{h}$ of type $D_m + D_{n-m}, 0 < m < n$, we have $D_k(\mathfrak{g}, \mathfrak{h}) = 4km\{(k-1)(n-m)(n+k-2)/(k + 2(n - 1))(k + 2(m - 1))(k + 2(n - m - 1))\}$. It follows that on level $k = 1$ every subalgebra whose simple components are of type D is conformally invariant. Also, the maximal subalgebra is conformally invariant on level $2 - k$.

E_6. $\mathfrak{h}$ is either indecomposable of type D_5 or decomposable of types $A_5 + A_1$ or $A_2 + A_2 + A_2$.   Thus, the possible values of $D_k(\mathfrak{g}, \mathfrak{h})$ are $k\{(84 + 33k)/(k + 12)(k + 8)\}$, $40k\{(k - 1)(k + 3)/(k + 12)(k + 6)(k + 2)\}$, $6k\{(8k - 21)/(k + 12)(k + 3)\}$. For $k = -3, 1$, the pair $(E_6, A_5 + A_1)$ is conformally invariant.

E_7. $\mathfrak{h}$ can be of the types $E_6, A_7, D_6 + A_1$ and $A_5 + A_2$.   The corresponding values of $D_k(\mathfrak{g}, \mathfrak{h})$ are $k\{(55k + 192)/(k + 18)(k + 12)\}$, $70k\{(k - 1)/(k + 8)(k + 18)\}$, $64k\{(k - 1)(k + 4)/(k + 18)(k + 10)(k + 2)\}$ and $90k\{(k - 1)(k + 4)/(k + 18)(k + 6)(k + 3)\}$. Note that we obtain conformally invariant pairs for $k = 1, -4$ and that subalgebras $D_6 + A_1$ and $A_5 + A_2$ produce conformally invariant pairs for the same levels.

E_8. $\mathfrak{h}$ is either indecomposable of the types A_8 and D_8 or decomposable of the types $E_7 + A_1, E_6 + A_2$ or $A_4 + A_4$.   The possible values of $D_k(\mathfrak{g}, \mathfrak{h})$ are $168k\{(k - 1)/(k + 30)(k + 9)\}$, $128k\{(k - 1)/(k + 30)(k + 14)\}$, $112k\{(k - 1)(k + 6)/(k + 30)(k + 18)(k + 2)\}$, $162k\{(k - 1)(k + 6)/(k + 30)(k + 12)(k + 3)\}$ and $200k\{(k - 1)/(k + 30)(k + 5)\}$. Note that all maximal subalgebras are conformally invariant for $k = 1$ and decomposable subalgebras with nonisomorphic components are also both conformally invariant on level $k = -6$.

F_4. $\mathfrak{h}$ can be of the types $B_4, A_2 + A_2$ or $C_3 + A_1$.   The possible values of $D_k(\mathfrak{g}, \mathfrak{h})$ are $8k\{(2k + 5)/(k + 9)(k + 7)\}$, $4k\{(20k^2 + 51k + 9)/(k + 9)(k + 3)(2k + 3)\}$ and $k\{(59k^2 + 61k - 70)/2(k + 9)(k + 4)(k + 1)\}$. There are no conformally invariant maximal regular subalgebras on integer levels (but there is a conformally invariant subalgebra on the half-integer level $-5/2$).

G_2. $\mathfrak{h}$ is either of the type A_2 or $A_1 + A_1$.   Thus, $D_k(\mathfrak{g}, \mathfrak{h})$ is either $2k\{(17k + 5)/3(k + 4)(k + 1)\}$ or $2k\{(15k^2 + 26k + 14)/(k + 4)(k + 2)(3k + 2)\}$. There are no conformally invariant maximal regular subalgebras.

Remark 6.1

(1) Our computations confirm the conclusion of [15] that for positive integer levels conformally invariant subalgebras exist only on level 1 (for a direct physical proof of this fact, see [12]). However, there are conformally invariant subalgebras on negative integer and half-integer levels.

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(2) The above discussion implies that conformally invariant subalgebras are rather rare. However, every maximal subalgebra of an algebra of type E_8 is conformally invariant. Such exceptional behavior is probably a consequence of some intrinsic property of affine algebras of type $E_8^{(1)}$.

6.3. Conformally invariant subalgebras of twisted affine algebras

There are analogs of the Sugawara and coset constructions of Virasoro algebras for twisted affine Kac–Moody algebras. Details can be found in, e.g., [11, section 3.4]. For our purposes it suffices to describe what central charges we can obtain from an algebra $\hat{\mathfrak{g}}(\sigma)$ of type $X_n^{(p)}$ and its subalgebra $\hat{\mathfrak{h}}(\sigma|_\mathfrak{h})$ of type $Y_m^{(q)}$. Here σ is an automorphism of $\mathfrak{g}$ that restricts to $\mathfrak{h}$. On the level k, we obtain the realization of the Virasoro algebra with the charge $D_k(\mathfrak{g})$, and corresponding to $\hat{\mathfrak{h}}(\sigma|_\mathfrak{h})$, another realization with the charge $D_{k'}(\mathfrak{h})$.

Recall that σ is completely determined by an $(m+1)$-tuple $(s_0, \ldots, s_l)$, where l is the rank of $\mathfrak{g}$: $\sigma(E_j) = \exp\left(2\pi \mathrm{i} s_j / \left(p \sum a_k s_k\right)\right) E_j$. Let $n_s(\mathfrak{g}) = \sum a_i s_i$ (here a_i are labels on the Dynkin diagram of $\hat{\mathfrak{g}}$). For the pair $\hat{\mathfrak{h}}(\tau) \subset \hat{\mathfrak{g}}(\sigma)$ set

$$r = \frac{q n_s(\mathfrak{h})}{p n_s(\mathfrak{g})} \qquad \text{and} \qquad D_k(\mathfrak{g}, \mathfrak{h}) = r D_k(\mathfrak{g}) - D_{k'}(\mathfrak{h}).$$

Below we will provide examples only for σ's induced by diagram automorphisms. Here $n_s(\mathfrak{g}) = n_s(\mathfrak{h}) = 1$.

$A_{2n}^{(2)}$. We consider the algebra $A_2^{(2)}$ separately. It has two distinct maximal subalgebras of the type $A_1^{(1)}$. The corresponding values of $D_k(\mathfrak{g}, \mathfrak{h})$ are $k\{(k-1)/(k+3)(k+2)\}$ and $k\{(13k-1)/2(k+3)(2k+1)\}$. Thus, there is a conformally invariant subalgebra on level 1.

For $n > 1$, $A_{2n}^{(2)}$ has subalgebras of the type $A_{2n-1}^{(2)}$, $B_n^{(1)}$, $A_{2m}^{(2)} + A_{2(n-m)-1}^{(2)}$, $D_m^{(1)} + A_{2n-2m}^{(2)}$, $D_3^{(2)} + A_{2n-4}^{(2)}$.

For a subalgebra of the type $A_{2n-1}^{(2)}$, $D_k(\mathfrak{g}, fh) = k\{(4n^2+1)(k+1) + 2n(4k+1)/2(k+2n+1)(k+n)\}$. A direct computation shows that no integer k makes this expression zero. Thus, a subalgebra of this type is never conformally invariant. For $n = 2$, the subalgebra obtained in the same manner has type $D_3^{(2)}$. Here $D_k(\mathfrak{g}, fh) = 5k\{(29k+37)/2(k+9)(k+2)\}$ and, again, the subalgebra is not conformally invariant.

For a subalgebra of the type $B_n^{(1)}$, $D_k(\mathfrak{g}, \mathfrak{h}) = kn\{(k-2n-3)/(k+2n+1)(k+2n-1)\}$. The subalgebra is conformally invariant on the level $k = 2n+3$.

For decomposable subalgebras of $A_{2n}^{(2)}$, we will write $D_k(\mathfrak{g}, \mathfrak{h})$ in terms of $N = 2n$ and $M = 2m$. For the type $A_{2m}^{(2)} + A_{2(n-m)-1}^{(2)}$, $D_k(\mathfrak{g}, \mathfrak{h}) = k[k(2(N-M)(M+N+MN) + k(4(M+M^2-N) - (M+N)^2) + (5N-3M)) - (N-1)(M-1) + M^2 + N^2 + (k+1)^2 + 1]/[(k+N+1)(k+M+1)(2k+N-M)]$. There are no conformally invariant subalgebras for small positive values of k.

For the type $D_m^{(1)} + A_{2n-2m}^{(2)}$, we have

$$D_k(\mathfrak{g}, \mathfrak{h}) = km \frac{2Nk(N-M+2k+1) + M(N-k-3k^2+2) + 5k^2 + 2k - 2}{2(k+N+1)(k+N-M+1)(k+m-1)}.$$

Again, there are no conformally invariant subalgebras for small positive values of k.

For the type $D_3^{(2)} + A_{2n-4}^{(2)}$, we have

$$D_k(\mathfrak{g}, \mathfrak{h}) = k \frac{2kN(8(k+N) - 7) + (N-1)^2 - 31k^2 - 18k + 12}{2(k+N+1)(k+N-3)(k+2)}.$$

The numerator is never zero mod 4, hence there are no conformally invariant subalgebras of this type.

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$A_{2n-1}^{(2)}$. Maximal regular subalgebras have the types $A_{n-1}^{(1)}$, $C_n^{(1)}$, $A_{2m-1}^{(2)} + A_{2(n-m)-1}^{(2)}$.

For the type $A_{n-1}^{(1)}$, $D_k(\mathfrak{g}, \mathfrak{h}) = 3nk\{(2nk+1)/2(k+2n)(2k+n)\}$. There are no conformally invariant pairs of these types.

For the type $C_n^{(1)}$, $D_k(\mathfrak{g}, \mathfrak{h}) = -(2n+1)k\{(4n^2 - 2n + 2 + (2n+1)k)/2(k+2n)(k+2n+2)\}$. This expression is zero only if $k \in (-2n-1, -2n-2)$, hence for integral k there are no conformally invariant subalgebras of this type.

For the type $A_{2m-1}^{(2)} + A_{2(n-m)-1}^{(2)}$, $D_k(\mathfrak{g}, \mathfrak{h})$ can be found from the expression for the subalgebra of type $A_m^{(1)} + A_{n-m-1}^{(1)}$ of the algebra $A_n^{(1)}$ above. Here we also conclude that there are no conformally invariant subalgebras on positive levels.

$D_{n+1}^{(2)}$. For a generic n, maximal regular subalgebras have the types $B_n^{(1)}$, $D_n^{(2)}$ and $D_{n-m}^{(1)} + D_{m+1}^{(2)}$.

For the type $B_n^{(1)}$, $D_k(\mathfrak{g}, \mathfrak{h}) = -k\{(2(2n^2 - n + 1) + (3n-1)k)/2(k+2n)(k+4n-2)\}$. $D_k(\mathfrak{g}, \mathfrak{h}) = 0$ for $k = -(4n^2 - 2n + 2)/(3n-1)$, hence $k \in (-4n/3 - 7/9, -4n/3 + 2/9)$. Depending on the value of $n \bmod 3$, k must be of the form $4n/3$, $(4n+1)/3$, $(4n+2)/3$. Only in the former case and only for $n = 3$, k makes $D_k(\mathfrak{g}, \mathfrak{h})$ zero. Thus, we have a conformally invariant subalgebra of $D_3^{(2)}$ on level -4.

For the type $D_n^{(2)}$, $D_k(\mathfrak{g}, \mathfrak{h}) = k\{(4n(n+k-1) + k - 2)/(k+2n)(k+2n-2)\}$. This expression is zero only if $k \in (n-2, n-1)$, hence there are no conformally invariant pairs of this type.

For the type $D_{n-m}^{(1)} + D_{m+1}^{(2)}$, $D_k(\mathfrak{g}, \mathfrak{h}) = -k(n-m)[8((n-m)(2mn+1+k(m+n)) - 1) + 2k(3n(1-2k) + m(6-5k) - 2mn) + 7k(2-k)]/[2(k+2n)(k+2m)(k+4n-4m-4)]$. We note that this subalgebra is not conformally invariant for $k = 1$.

An algebra of type $D_3^{(2)}$ also has a maximal subalgebra of type $C_2^{(1)}$. For this pair, $D_k(\mathfrak{g}, \mathfrak{h}) = -5k(5k+14)/2(k+4)(k+6)$.

$E_6^{(2)}$. A maximal regular subalgebra can have the types $C_4^{(1)}$, $F_4^{(1)}$, $A_5^{(2)} + A_1^{(1)}$. The respective values of $D_k(\mathfrak{g}, \mathfrak{h})$ are $-3k\{(11k+158)/(k+12)(k+10)\}$, $-13k\{(5k+42)/(k+12)(k+18)\}$ and $k\{(31k^2 + 4k - 672)/2(k+12)(k+6)(k+4)\}$. Thus, $C_4^{(1)}$ and $F_4^{(1)}$ subalgebras are never conformally invariant. The subalgebra of the type $A_5^{(2)} + A_1^{(1)}$ is not conformally invariant as well; direct computations also show that neither are its simple components.

$D_4^{(3)}$. Maximal regular subalgebras have the types $G_2^{(1)}$, $A_2^{(1)}$, $A_1^{(1)} + A_1^{(1)}$. The corresponding values of $D_k(\mathfrak{g}, \mathfrak{h})$ are $-28k\{(k+3)/(k+6)(k+12)\}$, $6k\{(k-1)/(k+6)(k+3)\}$ and $2k\{(k-4)/(k+6)(k+2)\}$. We obtain conformally invariant maximal subalgebras on the levels $k = -3, 1, 4$.

7. Regular non-indefinite subalgebras of hyperbolic Kac–Moody algebras

In this section, we apply the classification of regular subalgebras of affine Kac–Moody algebras to the investigation of regular subalgebras of hyperbolic Kac–Moody algebras.

Let Δ be a hyperbolic root system (i.e., a root system of a hyperbolic Kac–Moody algebra). Denote by W its Weyl group. Let $\Delta_1 \subset \Delta$ be a root subsystem containing only finite and affine indecomposable components. Then Δ_1 is a subsystem of some maximal affine or finite root subsystem Δ' of Δ. Root systems of hyperbolic Kac–Moody algebras are described by those Dynkin diagrams whose proper subdiagrams are of either finite or affine type. Maximal subdiagrams correspond exactly to maximal affine or finite root subsystems. Their Weyl groups can be obtained as groups generated by all but one reflections in the facets of the fundamental chamber of W.

Given a hyperbolic Kac–Moody algebra $\mathfrak{g}$, the description above allows us to classify all regular subalgebras containing no summands of indefinite type. More precisely, any such subalgebra can be obtained by the following procedure.

Take the Dynkin diagram Σ of the root system of $\mathfrak{g}$ and denote the number of its nodes by l. Consider all subdiagrams of Σ with exactly $l-1$ nodes. The corresponding subalgebras are either finite or affine. Now take all regular subalgebras of the algebras obtained. For this, we either rely on the results of section 5 (in the case of affine algebras) or the results of [2] (in the case of semisimple finite-dimensional algebras).

Acknowledgments

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